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Abstract
The deformation in a rate-dependent solid containing a pair of closely-spaced spherical voids is analytically determined. Results are obtained which are valid in the material that separates the cavities. The solid is assumed to be infinite in its extent, and is remotely subjected to a uniform stress field which is axially symmetric about the axes of the two voids; the cavities themselves remain traction free.

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Introduction

In this paper, the finite deformation and stress fields in the vicinity of a pair of spherical cavities in an all-around infinite rate-dependent solid are examined. The cavities remain traction-free as a uniform stress state is remotely applied to the surrounding material. The separation of the cavities is assumed to be much smaller than their radii. This assumption enables the use of a theory of plane stress to describe the deformation of the thin piece of material which separates the cavities. The theory used here is similar to that described by Adkins, et al (1954) for hyperelastic solids.

The problem dealt with here is of potential importance in connection with the rupture and failure of solids by the micromechanisms of void growth and coalescence. Thus, an understanding of the interaction of a pair of closely-spaced cavities serves to shed light on a possible crack-advance process. Such results might also eventually serve as useful guide to the development of constitutive models that accurately represent the behavior of porous solids at high levels of porosity. Existing models, such as that proposed by Gurson (1977) were designed to model materials that contain a sparse population of voids.

Within the linearized theory of elasticity, the problem of the interaction between a pair of spherical cavities in an isotropic, homogeneous solid was solved by Sternberg and Sadowsky (1952). They obtained solutions to the global problem in the form of infinite series, and did not need to assume that the voids were closely-spaced.

In the context of finite elasticity, single-void problems have been examined by several authors, including Green and Shield (1950), Ball (1982), and Horgan and Abeyaratne (1986). Single cavity problems in rate-dependent solids were examined by Abeyaratne (1988), who was particulary interested in the prediction of void nucleation. An analysis pertaining to contiguous voids in a class of hyperelastic solids was also recently done (Blume (1988)).

In the first section of this paper, some preliminaries on finite deformations of rate dependent solids are assembled. The particular constitutive class which will be used for the boundary-value problem at hand is introduced and discussed. In Section 2, the analysis of the two-cavity problem is carried out, and in the last section, the results are discussed.

1. Notation, Preliminaries on Finite Deformations of Rate-Dependent Solids

Throughout this paper, E_3 stands for three-dimensional Euclidean point space. Lower case and capital letters in boldface designate vectors and (second-order) tensors, respectively. The same letter in lightface appearing with one or more subscripts will signify the appropriate components of the vector or tensor in either cylindrical or Cartesian coordinates. If $\mathcal{A}$ is region in E_3, $f \in \mathcal{C}^m(\mathcal{A})$ (m=1,2,3,...) signifies that f is m times continuously differentiable on $\mathcal{A}$.

If $\mathcal{R}$ is the region in E_3 occupied by a solid in its reference configuration, a motion is a mapping $\hat{\mathbf{y}}$ described by

$$\mathbf{y} = \hat{\mathbf{y}}(\mathbf{x}, t) = \mathbf{x} + \mathbf{u}(\mathbf{x}, t) \text{ on } \mathcal{R}, \ (t > 0), \tag{1.1}$$

where $\mathbf{x}$ is the position vector of a generic point in $\mathcal{R}$, $\hat{\mathbf{y}}(\mathbf{x}, t)$ is the position vector of its deformation image at time t, $\mathbf{u}$ is the associated displacement field, and t is time. The region occupied by the body at time t will be called $\mathcal{R}_t$, and at each t, the mapping from $\mathcal{R}$ to $\mathcal{R}_t$ is assumed to be one-to-one. The motion $\hat{\mathbf{y}}$ of $\mathcal{R}$ is taken to be twice jointly continuously differentiable with respect to position and time.

The solids which will be dealt with here are incompressible and can thus sustain only locally volume-preserving deformations. The deformation (1.1) is locally-volume preserving if and only

$$J \equiv \det \mathbf{F} = 1 \text{ on } \mathcal{R}, \ (t > 0), \quad \mathbf{F} = \nabla \hat{\mathbf{y}}. \tag{1.2}$$

Here, $\mathbf{F}$ is the deformation gradient tensor and J is the Jacobian. The letters $\mathbf{C}$ and $\mathbf{G}$ stand for the right and left Cauchy-Green strain-tensor fields:

$$\mathbf{C} = \mathbf{F}^T\mathbf{F}, \quad \mathbf{G} = \mathbf{F}\mathbf{F}^T \text{ on } \mathcal{R}. \qquad (1.3)^1$$

Both $\mathbf{C}$ and $\mathbf{G}$ are symmetric, positive-definite tensor fields with common (positive) principal values. If $\mathbf{D}$ denotes the rate-of-deformation tensor, then

$$\mathbf{D}(\mathbf{y}, t) = \text{sym}\nabla\mathbf{v}(\mathbf{y}, t) \ \text{ on } \mathcal{R}_t, \quad t > 0 \tag{1.4}$$

[1] The superscript T indicates transposition.

in which $\mathbf{v}$ is the Eulerian velocity field:

$$\mathbf{v}(\mathbf{y},t) = \dot{\widehat{\mathbf{y}}}(\widehat{\mathbf{x}}(\mathbf{y},t),t) \;\text{ on } \mathcal{R}_t, \quad t > 0. \tag{1.5}$$

Here the dot denotes the derivative with respect to time ($\mathbf{x}$ fixed), and $\widehat{\mathbf{x}}(\mathbf{y},t)$ is the inverse of $\widehat{\mathbf{y}}(\mathbf{x},t)$ at time t: $\widehat{\mathbf{x}}(\widehat{\mathbf{y}}(\mathbf{x},t),t) = \mathbf{x}$ for all $\mathbf{x}$ in $\mathcal{R}$. The rate of deformation tensor is thus expressible as

$$\mathbf{D}(\widehat{\mathbf{y}}(\mathbf{x},t),t) = \frac{1}{2}\left\{\dot{\mathbf{F}}(\mathbf{x},t)\mathbf{F}^{-1}(\mathbf{x},t) + \mathbf{F}^{-T}(\mathbf{x},t)\dot{\mathbf{F}}^{T}(\mathbf{x},t)\right\}. \tag{1.6}$$

If $\boldsymbol{\sigma}$ is the Piola stress tensor accompanying the deformation at hand, equilibrium, in absence of body forces, demands that

$$\mathrm{div}\boldsymbol{\sigma} = \mathbf{0}, \; \boldsymbol{\sigma}\mathbf{F}^T = \mathbf{F}\boldsymbol{\sigma}^T \text{ on } \mathcal{R}. \tag{1.7}$$

In terms of the Cauchy stress field $\boldsymbol{\tau}$, which is linked to the Piola stress tensor through

$$\boldsymbol{\tau}(\mathbf{y},t) = \boldsymbol{\sigma}(\mathbf{x},t)\mathbf{F}^T(\mathbf{x},t), \; \mathbf{y} = \widehat{\mathbf{y}}(\mathbf{x}), \tag{1.8}$$

equilibrium may be written as

$$\mathrm{div}\boldsymbol{\tau}(\mathbf{y},t) = \mathbf{0}, \;\; \boldsymbol{\tau}(\mathbf{y},t) = \boldsymbol{\tau}^T(\mathbf{y},t) \quad \text{for all } \mathbf{y} \in \mathcal{R}_t \tag{1.9}$$

Note that *inertial effects are neglected* in the analysis carried out here.

The stress response of materials at hand is assumed to be well described by a purely rate-dependent, power-law type constitutive law. In particular, if ν and γ are material constants with $\nu > 0$, then

$$\boldsymbol{\tau} = -p\mathbf{1} + \nu d^{\gamma}\mathbf{D} \text{ on } \mathcal{R}_t. \tag{1.10}$$

where $p = p(\mathbf{y},t)$ is the pressure field which accommodates the constraint of incompressibility, $\mathbf{1}$ is the identity tensor, and

$$d = |\mathbf{D}| = \sqrt{D_{ij}D_{ij}} \text{ on } \mathcal{R}_t. \tag{1.11}$$

Note that the second of (1.9) is satisfied automatically. On account of (1.8), (1.3), and (1.6), the Piola stress for such materials may be written as

$$\boldsymbol{\sigma} = -p\mathbf{F}^{-T} + \frac{1}{2}\nu d^{\gamma}\left\{\dot{\mathbf{F}}\mathbf{C}^{-1} - \left(\mathbf{F}^{-T}\right)^{\cdot}\right\} \text{ on } \mathcal{R}. \tag{1.12}$$

In order to understand the behavior of such a solid, consider a state of *uniaxial stress*. If the body is in a state of uniaxial stress, then only one stress component, say τ_{11} is nonzero and depends only on time; all others vanish identically in $\mathcal{R}$. One motion and pressure field which corresponds to such a state is

$$\begin{gathered}\widehat{y}_1(\mathbf{x},t) = \lambda(t)x_1, \;\; \widehat{y}_2(\mathbf{x},t) = \lambda^{-1/2}(t)x_2, \;\; \widehat{y}_3(\mathbf{x},t) = \lambda^{-1/2}(t)x_3 \\ p(\mathbf{x},t) = -\frac{1}{2}\nu\left(\frac{3}{2}\right)^{\gamma/2}\left(\frac{\dot{\lambda}}{\lambda}\right)^{\gamma+1}.\end{gathered} \tag{1.13}$$

Here $\lambda > 0$ for $t > 0$, and λ is a twice continuously differentiable function of time. One has

$$\tau_{11} = \bar{\nu}\left(\frac{\dot{\lambda}}{\lambda}\right)^{\gamma+1} \quad \bar{\nu} = \nu\left(\frac{3}{2}\right)^{\gamma/2+1}. \tag{1.14}$$

Further,

$$\lambda(t) = c\exp\left\{\int_0^t \left(\frac{\tau_{11}(s)}{\bar{\nu}}\right)^{\frac{1}{1+\gamma}} ds\right\}, \quad c \text{ constant}. \tag{1.15}$$

For $\tau_{11}(t) = \tau t$, (τ constant)

$$\lambda(t) = c\exp\left\{\left(\frac{\tau}{\bar{\nu}}\right)^{\frac{1}{\gamma+1}} \frac{\gamma+1}{\gamma+2}\, t^{\frac{\gamma+2}{\gamma+1}}\right\}. \tag{1.16}$$

The response is shown in Figure 1.

2. The Two-Cavity Problem

If $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ are base vectors of a Cartesian coordinate frame, the solid initially occupies the unbounded region $\mathcal{R}$, described by

$$\mathcal{R} = \{\mathbf{x} |\ |\mathbf{x} - \delta \mathbf{e}_3| > 1,\ |\mathbf{x} + \delta \mathbf{e}_3| > 1\}. \tag{2.1}$$

The coordinate system is chosen so that the holes have unit radii; their centers are located at $x_3 = \pm\delta$, with $\delta > 1$. The voids are thus separated by a distance of $2(\delta - 1)$. In what follows, it is assumed that

$$1 < \delta \ll 2; \tag{2.2}$$

this assures that the cavities are closely-spaced. In order that the cavity surfaces remain traction free, the nominal traction vector must vanish there:

$$\boldsymbol{\sigma}\mathbf{n} = \mathbf{0} \text{ on } \delta\mathcal{R} \equiv \{\mathbf{x} |\ |\mathbf{x} \pm \delta \mathbf{e}_3| = 1\}, \tag{2.3}$$

or equivalently, the Cauchy traction vanishes on the deformed cavity walls, whence

$$\boldsymbol{\tau}\overset{*}{\mathbf{n}} = \mathbf{0} \text{ on } \partial\mathcal{R}_t. \tag{2.4}$$

In the preceding expressions, $\mathbf{n}$ is the outer unit normal vector to $\partial\mathcal{R}$; $\overset{*}{\mathbf{n}}$ is the outer unit normal vector to the cavity surfaces in the current configuration. At large distances from the voids, the solid is subjected to a uniform state of stress which has axial symmetry about the x_3-axis as well as symmetry about the plane $x_3 = 0$. The problem geometry is shown in Figure 2.

The main purpose of this analysis is to characterize the deformation in the material which separates the voids. For moderate values of the remotely applied stress field, and for large times, it appears reasonable to assume that the largest strains and strain-rates occur in this region, and that rest of the solid might well undergo infinitesimal deformations at low rates.

Thus, consider the region $\mathcal{R}_*$, that separates the voids in the undeformed configuration (Figure 3):

$$\mathcal{R}_* = \{\mathbf{x}|\ r < a,\ -h(r) < x_3 < h(r)\}\,,\quad r = \sqrt{x_1^2 + x_2^2},\quad h(r) = \delta - \sqrt{1 - r^2}, \tag{2.5}$$

and $2(\delta - 1) < a < 1$ is a real number. Assume also that the (axially symmetric) remote load in the global two-cavity problem produces normal and shear traction distributions $\overset{*}{\sigma}_n(x_3, t)$ and $\overset{*}{\sigma}_s(x_3, t)$ on the lateral boundary of the disk $\mathcal{R}_*$, so that the components of the Piola stress in cylindrical coordinates (r, θ, x_3) obey

$$\left.\begin{aligned} \sigma_{rr}(r, x_3, t) &= \overset{*}{\sigma}_n(x_3, t), \\ \sigma_{r3}(r, x_3, t) &= \overset{*}{\sigma}_s(x_3, t), \end{aligned}\right. \quad r = a,\quad -h(a) < x_3 < h(a),\quad t > 0. \tag{2.6}$$

The symmetry of the problem mandates that

$$\overset{*}{\sigma}_n(x_3, t) = \overset{*}{\sigma}_n(-x_3, t)\quad \overset{*}{\sigma}_s(x_3, t) = -\overset{*}{\sigma}_s(-x_3, t),\quad -h(a) < x_3 < h(a). \tag{2.7}$$

The faces of the disk $x_3 = \pm h(r)$ $(0 \leq r < a)$ are traction-free. In terms of the cylindrical components of Piola stress, this last condition is equivalent to

$$\left.\begin{aligned} &\pm\, \sigma_{rr}(r, x_3, t)h'(r) - \sigma_{r3}(r, x_3, t) = 0, \\ &\pm\, \sigma_{3r}(r, x_3, t)h'(r) - \sigma_{33}(r, x_3, t) = 0 \end{aligned}\right. \quad \text{on}\ \ x_3 = \pm h(r),\ h' \equiv \frac{dh}{dr},\ (0 \leq r < a). \tag{2.8}$$

At this point, it is useful to recall that for axially symmetric motions, the equilibrium equations (1.7) in terms of the cylindrical coordinates (r, θ, x_3) reduce to

$$\begin{aligned} \frac{\partial}{\partial r}\sigma_{rr} + \frac{1}{r}(\sigma_{rr} - \sigma_{\theta\theta}) + \frac{\partial}{\partial x_3}\sigma_{r3} &= 0, \\ \frac{\partial}{\partial r}\sigma_{3r} + \frac{\partial}{\partial x_3}\sigma_{33} + \frac{1}{r}\sigma_{r3} &= 0. \end{aligned} \tag{2.9}$$

Using plane-stress approximative assumptions, the motion in the disk $\mathcal{R}_*$ induced by the edge-load distributions $\overset{*}{\sigma}_n$ and $\overset{*}{\sigma}_s$ will be calculated. Owing to the symmetries inherent in the problem, any motion $\hat{\mathbf{y}}$ of $\mathcal{R}$ (and hence also of the subregion $\mathcal{R}_*$ of $\mathcal{R}$) has the form

$$\left.\begin{aligned} \hat{y}_\alpha(\mathbf{x}, t) &= R(r, x_3, t)x_\alpha/r,\quad \hat{y}_3(r, x_3, t) = Z(r, x_3, t), \\ R(r, x_3, t) &= R(r, -x_3, t),\quad Z(r, x_3, t) = -Z(r, -x_3, t), \end{aligned}\right. \quad (\alpha = 1, 2) \tag{2.10}$$

with $R = 0$ when $r = 0$, but R and Z nonvanishing and twice continuously differentiable for $(0 < r < a)$. Further, the nominal stress components obey

$$\sigma_{rr}(r,x_3,t) = \sigma_{rr}(r,-x_3,t), \quad \sigma_{\theta\theta}(r,x_3,t) = \sigma_{\theta\theta}(r,-x_3,t), \; \sigma_{33}(r,x_3,t) = \sigma_{33}(r,-x_3,t),$$
$$\sigma_{r3}(r,x_3,t) = -\sigma_{r3}(r,-x_3,t), \; \sigma_{3r}(r,x_3,t) = -\sigma_{3r}(r,-x_3,t); \tag{2.11}$$

all other stress components vanish.

Let Π stand for the midplane $(x_3 = 0)$ of $\mathcal{R}_*$, and if f is a function defined on $\mathcal{R}_*$, $\mathring{f}$ will be used to denote its restriction to Π. The aim of the following is to characterize the deformation parameters $\mathring{R}$ and $\partial Z(r,0,t)/\partial x_3 \equiv \mathring{\lambda}(r,t)$, along with the stress components $\mathring{\sigma}_{rr}$ and $\mathring{\sigma}_{\theta\theta}$ by assuming that

$$\mathring{\sigma}_{33} = 0 \text{ on } \Pi, \quad \frac{\partial}{\partial x_3}\sigma_{rr} = \frac{\partial}{\partial x_3}\sigma_{\theta\theta} = 0 \text{ on } \mathcal{R}_*. \tag{2.12}$$

These may be justified on the basis of the geometry of the problem, the second of (2.8), and the second of (2.9).

Integrating the first of the equilibrium equations (2.9) across the thickness of $\mathcal{R}_*$ leads to

$$\int_{-h(r)}^{h(r)} \left\{ \frac{\partial}{\partial r}\sigma_{rr} + \frac{1}{r}\left(\sigma_{rr} - \sigma_{\theta\theta}\right) \right\} dx_3 + \sigma_{r3}\Big|_{x_3=-h(r)}^{x_3=h(r)} = 0. \tag{2.13}$$

The first of (2.8) and the assumptions (2.12) enable

$$\frac{\partial}{\partial r}\mathring{\sigma}_{rr} + \frac{1}{r}\left(\mathring{\sigma}_{rr} - \mathring{\sigma}_{\theta\theta}\right) + \frac{h'}{h}\mathring{\sigma}_{rr} = 0 \text{ on } \Pi. \tag{2.14}$$

The boundary condition in the first of (2.6) corresponds under the present circumstance to the stipulation that

$$\mathring{\sigma}_{rr}(a,t) = \overset{*}{\sigma}_n(a,0,t). \tag{2.15}$$

Note that the integral of the shear traction $\overset{*}{\sigma}_s$ across the thickness vanishes by virtue of the symmetry condition that appears in the second of (2.7).

With a view towards expressing the stresses $\mathring{\sigma}_{rr}$ and $\mathring{\sigma}_{\theta\theta}$ in terms of the deformation parameters $\mathring{R}$ and $\mathring{\lambda}$, note that (2.10) implies that the nonzero cylindrical components of the deformation gradient tensor $\mathring{\mathbf{F}}$ are

$$\mathring{F}_{rr} = \mathring{R}_{,r}, \quad \mathring{F}_{\theta\theta} = \mathring{R}/r, \quad \mathring{F}_{33} = \mathring{\lambda} \equiv \partial Z(r,0,t)/\partial x_3 \text{ on } \Pi. \tag{2.16}$$

Here and in what follows, $\partial\mathring{R}/\partial r \equiv \mathring{R}_{,r}$, while $\partial\mathring{R}/\partial t \equiv \mathring{R}_{,t}$.

Incompressibility demands that $\det\mathring{\mathbf{F}} = 1$ on Π, whence

$$\mathring{\lambda} = \frac{r}{\mathring{R}_{,r}\mathring{R}} \text{ on } \Pi. \tag{2.17}$$

The stress components are now found from (1.12) to be

$$\begin{gathered}\mathring{\sigma}_{rr} = \left\{-\mathring{p} + \nu d^\gamma \frac{\mathring{R}_{,rt}}{\mathring{R}_{,r}}\right\}\frac{1}{\mathring{R}_{,r}}, \quad \mathring{\sigma}_{\theta\theta} = \left\{-\mathring{p} + \nu d^\gamma \frac{\mathring{R}_{,t}}{\mathring{R}}\right\}\frac{r}{\mathring{R}}, \\ \mathring{\sigma}_{33} = \left\{-\mathring{p} - \nu d^\gamma \left[\frac{\mathring{R}_{,t}}{\mathring{R}} + \frac{\mathring{R}_{,rt}}{\mathring{R}_{,r}}\right]\right\}\frac{\mathring{R}\mathring{R}_{,r}}{r},\end{gathered} \tag{2.18}$$

and

$$d = \sqrt{2\left(\frac{\mathring{R}_{,rt}}{\mathring{R}_{,r}}^{2} + \left(\frac{\mathring{R}_{,t}}{\mathring{R}}\right)^2 + \frac{\mathring{R}_{,rt}\mathring{R}_{,t}}{\mathring{R}_{,r}\mathring{R}}\right)} \text{ on } \Pi. \tag{2.19}$$

The last of (2.18), in conjunction with the plane stress assumption appearing in the first of (2.12) gives

$$\mathring{p} = -\nu d^\gamma \left[\frac{\mathring{R}_{,t}}{\mathring{R}} + \frac{\mathring{R}_{,rt}}{\mathring{R}_{,r}}\right] \text{ on } \Pi, \tag{2.20}$$

whence

$$\mathring{\sigma}_{rr} = \nu d^\gamma \left[\frac{\mathring{R}_{,t}}{\mathring{R}} + 2\frac{\mathring{R}_{,rt}}{\mathring{R}_{,r}}\right]\frac{1}{\mathring{R}_{,r}}, \quad \mathring{\sigma}_{\theta\theta} = \nu d^\gamma \left[2\frac{\mathring{R}_{,t}}{\mathring{R}} + \frac{\mathring{R}_{,rt}}{\mathring{R}_{,r}}\right]\frac{2}{\mathring{R}}. \tag{2.21}$$

On entering (2.20) into (2.14), one is led to a nonlinear, homogeneous partial differential equation for $\mathring{R}$ with nonconstant coefficients. This equation involves the second derivative of $\mathring{R}$ with respect to r, and its first derivative with respect to time. The boundary condition (2.15) requires that $\mathring{R}$ conform to

$$\frac{\nu d^\gamma}{\mathring{R}_{,r}}\left[\frac{\mathring{R}_{,t}}{\mathring{R}} + 2\frac{\mathring{R}_{,rt}}{\mathring{R}_{,r}}\right]_{r=a} = \overset{*}{\sigma}_n(a,0,t). \tag{2.22}$$

Of particular physical interest in the problem at hand is the large-time behavior of $\mathring{R}$. This is perhaps when the neglection of inertial effects is most valid. It is reasonable to assume that $\mathring{R}$ is of the form

$$\mathring{R}(r,t) = A(t)q(r), \quad 0 \leq r < a, \quad t > 0, \tag{2.23}$$

with A and q twice differentiable. In this case, (2.19) gives

$$d = \sqrt{6}\frac{\dot{A}}{A}, \quad \dot{A} \equiv \frac{dA}{dt}, \tag{2.24}$$

and (2.21) leads to

$$\mathring{\sigma}_{rr}(r,t) = \frac{m(t)}{q'(r)}, \quad \mathring{\sigma}_{\theta\theta} = r\frac{m(t)}{q(r)} \text{ on } \Pi, \quad q' \equiv \frac{dq}{dr}, \tag{2.25}$$

in which

$$m = 3\nu 6^{\gamma/2}\left\{\frac{\dot{A}}{A}\right\}^{\gamma+1}. \tag{2.26}$$

Substitution from (2.25) into the equilibrium equation (2.14) and rearranging leads to the following ordinary differential equation for q:

$$-r\left[qq'' + (q')^2\right] + qq' + r\frac{h'}{h}qq' = 0, \quad 0 < r < a. \tag{2.27}$$

If one sets

$$v(r) = q(r)q'(r), \quad 0 \leq r \leq a, \tag{2.28}$$

then (2.27) reduces to the first order equation

$$v' - \left(\frac{1}{r} + \frac{h'}{h}\right)v = 0, \tag{2.29}$$

the solution to which is given by

$$v(r) = crh(r). \tag{2.30}$$

Here, c is a constant. Bearing in mind that $\mathring{R}(0,t) = 0$, one finds on integrating (2.28) that

$$q(r) = \sqrt{2\int_0^r v(\rho)d\rho}. \tag{2.31}$$

Appealing to the definition of h in the last of (2.5) and the expression for $\mathring{R}$ in (2.23), one has

$$\mathring{R}(r,t) = A(t)\sqrt{\delta r^2 + \frac{2}{3}\left(1-r^2\right)^{3/2} - \frac{2}{3}}. \tag{2.32}$$

Plots of $\mathring{R}$ versus r for fixed t are shown in Figure 4.

The function $A(t)$ is related to $\overset{*}{\sigma}_n(a,0,t)$ through (2.22), which now gives

$$\begin{aligned} A(t) &= \exp\left\{ \left[\frac{q'(a)}{3\nu}6^{-\gamma/2}\right]^{\frac{1}{\gamma+1}} \int_0^t \overset{*}{\sigma}_n^{\frac{1}{\gamma+1}}(a,o,s)ds \right\} \\ q'(a) &= 2a\frac{\delta - \sqrt{1-a^2}}{\sqrt{\delta a^2 + \frac{2}{3}(1-a^2)^{3/2} - \frac{2}{3}}}. \end{aligned} \tag{2.33}$$

If $\overset{*}{\sigma}_n$ is time-independent, $A(t)$ is an exponential function of time, while if $\overset{*}{\sigma}_n$ is a linear function of time, $A(t)$ is a constant multiple of the function $\lambda(t)$ in (1.16), which is plotted in Figure 1.

According to (2.32), (2.17), the out-of-plane stretch $\mathring{\lambda}$ is given by

$$\mathring{\lambda}(r,t) = A^{-2}(t)\frac{1}{\delta - \sqrt{1-r^2}}, \tag{2.34}$$

which, on account of the last of (2.5) may be written as

$$\mathring{\lambda}(r,t) = \frac{1}{A^2(t)h(r)}. \tag{2.35}$$

The boundary $x_3 = h(r)$ is thus approximately mapped to the plane $y_3 = Z = A^{-2}(t)$ and the material in between the spheres is flattening. Figure 5 shows $\mathring{\lambda}$ versus r, while Figure 6 shows the deformed disks for various values of $A(t)$ at fixed t. In particular, note that

$$\mathring{\lambda}(0,t) = \frac{1}{A^2(t)\,(\delta-1)}, \tag{2.36}$$

is the maximum value of $\mathring{\lambda}$ at fixed t; the minimum occurs on the edge of the disk ($r = a$).

As far as the stresses are concerned, the Nominal stresses and Cauchy stresses are found from (2.32), (2.21), (1.8):

$$\begin{gathered} \mathring{\sigma}_{rr} = m(t)\frac{\delta r - r\sqrt{1-r^2}}{\sqrt{\delta r^2 + \frac{2}{3}\left(1-r^2\right)^{3/2} - \frac{2}{3}}}, \quad \mathring{\sigma}_{\theta\theta} = \frac{m(t)r}{\sqrt{\delta r^2 + \frac{2}{3}\left(1-r^2\right)^{3/2} - \frac{2}{3}}} \\ \tau_{rr} = \tau_{\theta\theta} = m(t), \end{gathered} \tag{2.37}$$

with m given by (2.26), (2.32). Note that the Cauchy stresses are constant.

3. Discussion

In this analysis, it was assumed that the material between the voids does not rupture. This was reflected in the stipulation that R vanishes at $r = 0$. If this requirement is removed and replaced with $R > 0$ at $r = 0$, the possibility of a hole opening at the center of the disk is admitted. In this case, in order that the newly-formed hole be traction-free, one demands that

$$\sigma_{rr} = \sigma_{3r} = 0 \text{ if } r = 0. \tag{3.1}$$

Pursuing the plane stress analysis exactly as before leads one to

$$\overset{\circ}{R}(r,t) = A(t)\sqrt{\delta r^2 + \frac{2}{3}\left(1 - r^2\right)^{3/2} - b}, \tag{3.2}$$

with $A > 0$ a function of time and b constant, $b < 2/3$. The boundary condition (2.22) must hold again in this case, but in place of the requirement that $\overset{\circ}{R}(0,t) = 0$, one has from the first of (3.1) that

$$\overset{\circ}{\sigma}_{rr}(0,t) = \frac{\nu d^\gamma}{\overset{\circ}{R}_{,r}}\left[\frac{\overset{\circ}{R}_{,t}}{\overset{\circ}{R}} + 2\frac{\overset{\circ}{R}_{,rt}}{\overset{\circ}{R}_{,r}}\right]_{r=0} = 0. \tag{3.3}$$

The second of (3.1) is satisfied in average across the thickness by virtue of symmetry. However, the condition (3.3) can be met only if $b = 0$ so that plane stress deformations of the form (2.23) with holes emerging at the origin are not present.

Although no restrictions (other than symmetry conditions) were put on the remote stress field in the global problem, the plane stress assumptions are perhaps more believable in situations in which the disk $\mathcal{R}_*$ is itself in a tensile field (so that it is *thinning*); presumably this situation would correspond to remote tensile loading.

The validity of the plane stress assumptions is increased if boundary of the disk $\mathcal{R}_*$ is chosen so that the magnitude of the shear traction $\overset{*}{\sigma}_s(x_3,t)$ is minimized across the thickness of $\mathcal{R}_*$. As yet, a means of connecting the tractions $\overset{*}{\sigma}_n$ and $\overset{*}{\sigma}_s$ with the remote loading is unavailable; this is the subject of future work. The effects of inertia on the deformation of $\mathcal{R}_*$ are also being examined; plane stress may not be a valid assumption in the dynamic case.

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Figure Captions

Figure 1: Response of the power law viscous solids in uniaxial stress;

$$\lambda_*(t) = \lambda(t)\exp\left\{-\left(\frac{\tau}{\bar{\nu}}\right)^{1/(\gamma+1)}\frac{\gamma+1}{\gamma+2}\right\}.$$

Figure 2: Problem geometry.

Figure 3: The region $\mathcal{R}_*$.

Figure 4: $\mathring{R}$ versus r.

Figure 5: The out-of-plane stretch $\mathring{\lambda}$ versus r.

Figure 6: Deformed disks.

Figure 1

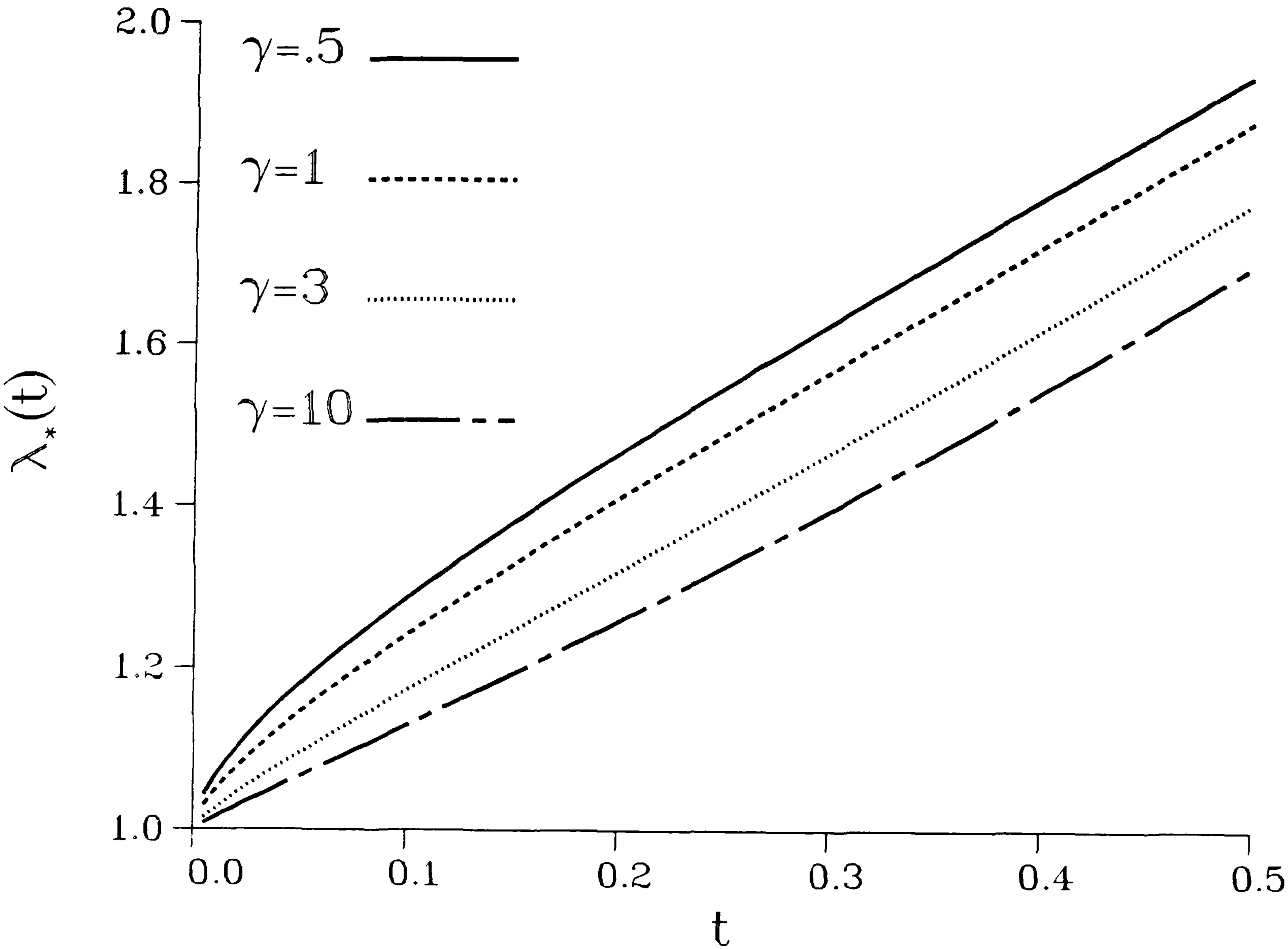

Figure 2

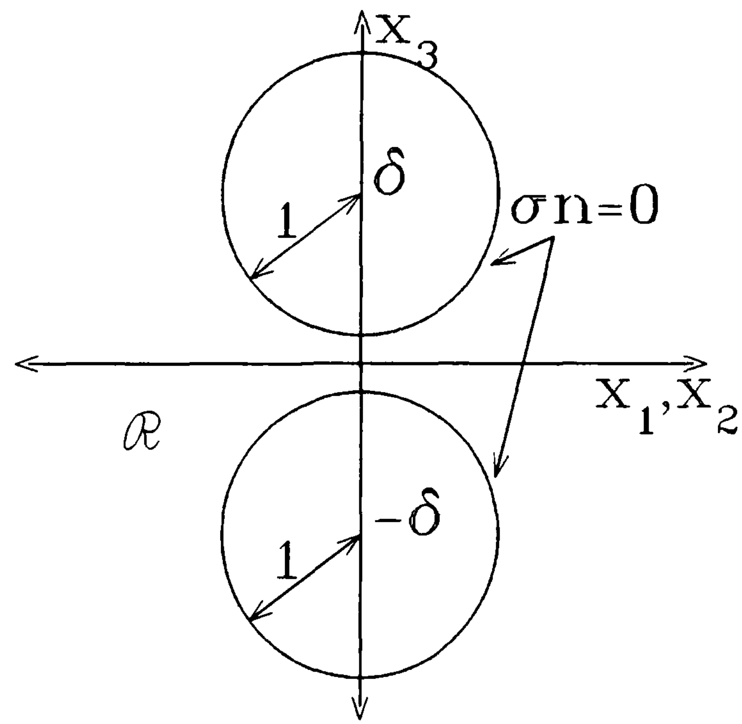

Figure 3

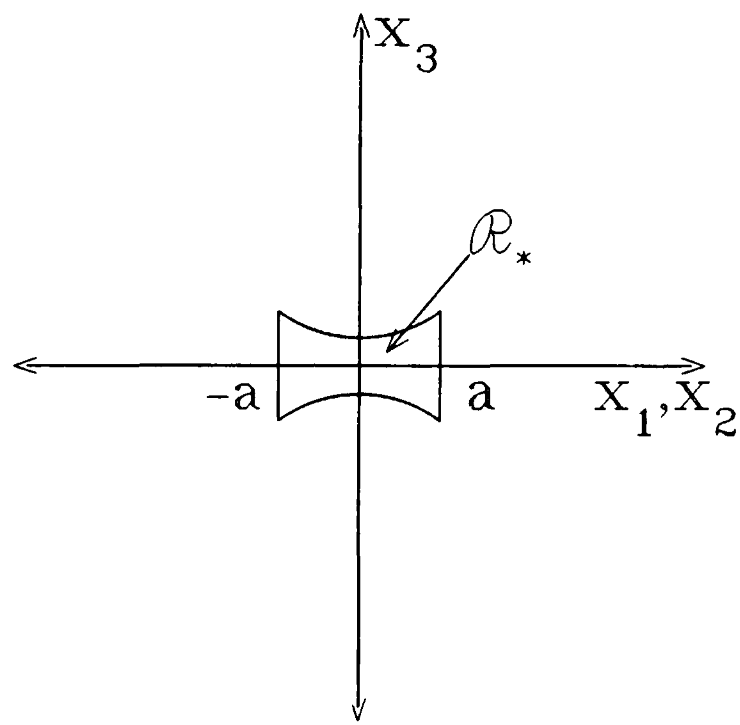

Figure 4

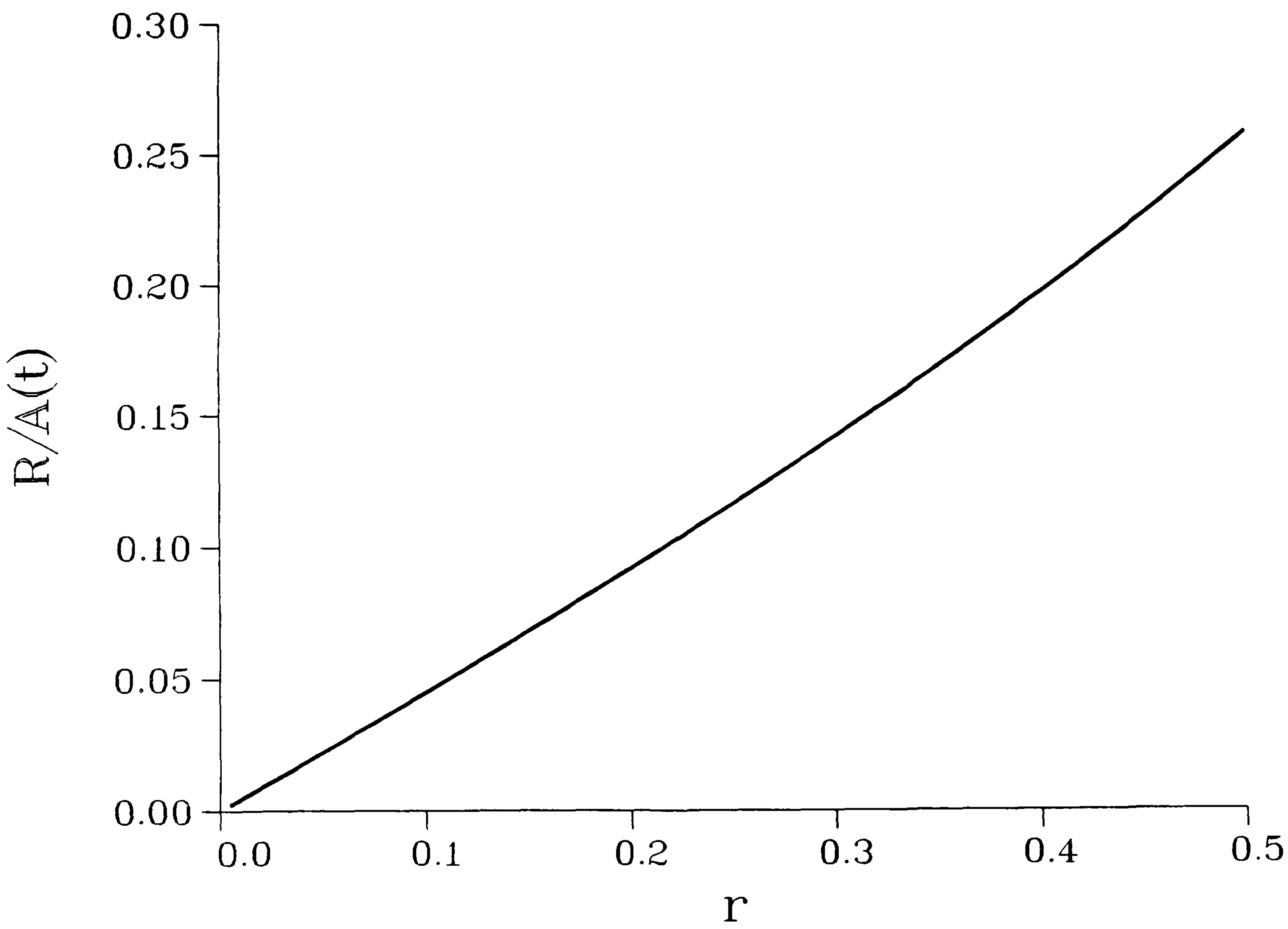

Figure 5

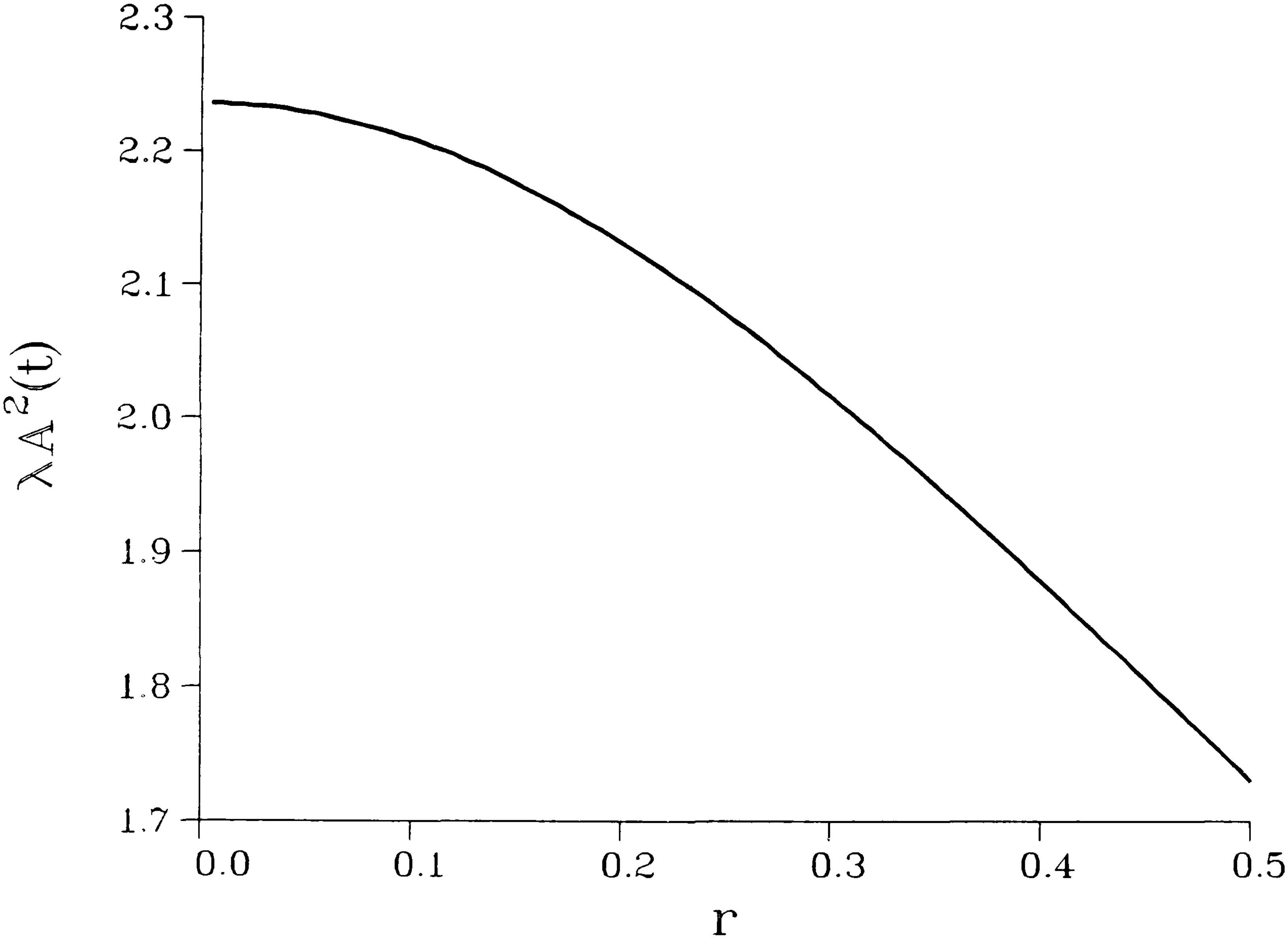

Figure 6